Sheet (2)

1. A hypothetical isotropic antenna is radiating in free-space. At a distance of 100 m from the antenna, the total electric field (E_θ) is measured to be 5 V/m. Find the
(a) Power density (W_{rad})
(b) Power radiated (P_{rad})

(a) $\underline{W}_{rad} = \frac{1}{2}[\underline{E} \times \underline{H}^*] = \frac{E^2}{2\eta}\hat{a}_r = \frac{5^2 \hat{a}_r}{2(120\pi)} = 0.03315\ \hat{a}_r$ watts/m²

(b) $P_{rad} = \oint_S W_{rad}\, dS = \int_0^{2\pi}\int_0^{\pi} (0.03315)(r^2 \sin\theta\, d\theta\, d\phi)$

$= \int_0^{2\pi}\int_0^{\pi} (0.03315)(100)^2 \sin\theta\, d\theta\, d\phi$

$= 2\pi\,(0.03315)(100)^2 \cdot \int_0^{\pi} \sin\theta\, d\theta = 2\pi(0.03315)(100)^2 \cdot 2$

$= 4165.75$ watts

2. The power radiated by a lossless antenna is 10 watts. The directional characteristics of the antenna are represented by the radiation intensity of
$U=B_o\cos^3\theta$ (watts/unit solid angle) ($0\le\theta\le\pi/2$, $0\le\varphi\le2\pi$)
Find the
(a) Maximum power density (in watts/square meter) at a distance of 1,000 m (assume far-field distance). Specify the angle where this occurs.
(b) Exact and approximate beam solid angle Ω_A.
(c) Directivity, exact and approximate, of the antenna (dimensionless and in dB).
(d) Gain, exact and approximate, of the antenna (dimensionless and in dB).

$$U = B_0 \cos^3\theta$$

(a) $$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U \sin\theta \, d\theta \, d\phi = B_0 \int_0^{2\pi}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta \, d\phi$$

$$= 2\pi B_0 \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$$

$$P_{rad} = 2\pi B_0 \left(-\frac{\cos^4\theta}{4}\right)\Bigg|_0^{\pi/2} = \frac{\pi}{2} B_0 = 10 \Rightarrow B_0 = \frac{20}{\pi} = 6.3662$$

$$U = 6.3662 \cos^3\theta$$

$$W = \frac{U}{r^2} = \frac{6.3662}{r^2}\cos^3\theta = \frac{6.3662}{(10^3)^2} \cdot \cos^3\theta = 6.3662 \times 10^{-6} \cos^3\theta$$

$$W\big|_{max} = 6.3662 \times 10^{-6} \cdot \cos^3\theta\Big|_{max} = 6.3662 \times 10^{-6} \text{ Watts}/m^2$$

(b) $$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\,(6.3662)}{10} = 8 = 9\,dB$$

(c) $$G_0 = e_t D_0 = 8 = 9\,dB$$

3. In target-search ground-mapping radars it is desirable to have echo power received from a target, of constant cross section, to be independent of its range. For one such application, the desirable radiation intensity of the antenna is given by

$$U(\theta,\phi) = \left\{ \begin{array}{cc} 1 & 0^\circ \le \theta < 20^\circ \\ 0.342\csc(\theta) & 20^\circ \le \theta < 60^\circ \\ 0 & 60^\circ \le \theta \le 180^\circ \end{array} \right\} 0^\circ \le \phi \le 360^\circ$$

Find the directivity (in dB) using the exact formula.

$$U(\theta,\phi)=\begin{cases} 1 & 0^\circ \leq \theta \leq 20^\circ \\ 0.342\ \csc(\theta) & 20^\circ \leq \theta \leq 60^\circ \\ 0 & 60^\circ \leq \theta \leq 180^\circ \end{cases} \quad 0^\circ \leq \phi \leq 360^\circ$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi\left[\int_0^{20^\circ}\sin\theta\, d\theta + \int_{20^\circ}^{60^\circ} 0.342\ \csc(\theta) \times \sin\theta\, d\theta\right] = 2\pi\left\{-\cos\theta\Big|_0^{\pi/9} + 0.342\cdot\theta\Big|_{\pi/9}^{\pi/3}\right\}$$

$$= 2\pi\left\{\left[-\cos\left(\frac{\pi}{9}\right)+1\right] + 0.342\left(\frac{\pi}{3}-\frac{\pi}{9}\right)\right\}$$

$$= 2\pi\left\{\left[-0.93969+1\right] + 0.342\pi\left(\frac{2}{9}\right)\right\}$$

$$= 2\pi\left\{0.06031 + 0.23876\right\} = 1.87912$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(1)}{1.87912} = 6.68737 = 8.25255\text{ dB.}$$

4. The normalized radiation intensity of a given antenna is given by
(a) U=sinθsinφ, (b) U=sinθsin^{2}φ, (C) U=sin^{2}θsin^{3}φ
The intensity exists only in the $0\leq\theta\leq\pi$, $0\leq\varphi\leq\pi$ region, and it is zero elsewhere. Find the
(a) Exact directivity (dimensionless and in dB).
(b) Azimuthal and elevation plane half-power beam widths (in degrees).

$$D_o = \frac{4\pi\, U_{max}}{P_{rad}}$$

(a) $U = \sin\theta \sin\phi$ for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant \pi$

$U|_{max} = 1$ and it occurs when $\theta = \phi = \pi/2$.

$$P_{rad} = \int_0^{\pi}\int_0^{\pi} U \sin\theta\, d\theta\, d\phi = \int_0^{\pi} \sin\phi\, d\phi \int_0^{\pi} \sin^2\theta\, d\theta = 2\left(\frac{\pi}{2}\right) = \pi.$$

Thus $D_o = \frac{4\pi(1)}{\pi} = 4 = 6.02\,dB$

The half-power beamwidths are equal to

HPBW (az.) $= 2[90° - \sin^{-1}(1/2)] = 2(90° - 30°) = 120°$

HPBW (el.) $= 2[90° - \sin^{-1}(1/2)] = 2(90° - 30°) = 120°$

In a similar manner, it can be shown that for

(b) $U = \sin\theta \sin^2\phi \Rightarrow D_o = 5.09 = 7.07\,dB$

HPBW (el.) $= 120°$, HPBW (az.) $= 90°$

(C)

$U = \sin^2\theta \sin^3\phi \Rightarrow D_o = 9\pi/4 = 7.07 = 8.49\,dB$

HPBW (el.) $= 90°$, HPBW (az.) $= 74.93°$

5. Find the directivity (dimensionless and in dB) for the antenna of Problem 4 using Kraus' approximate formula.

(a)

$U = \sin\theta \cdot \sin\phi$; (a) $D_o \simeq \frac{41\,253}{\Theta_{1d}\,\Theta_{2d}} = \frac{41\,253}{120\,(120)} = 2.86 = 4.57\,dB$

(b) $D_o \simeq 3.82 = 5.82\,dB$

(c) $D_o \simeq 6.12 = 7.87\,dB$

6. The normalized radiation intensity of an antenna is rotationally symmetric in φ, and it is represented by

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$$U = \begin{cases} 1 & 0^\circ \le \theta < 30^\circ \\ 0.5 & 30^\circ \le \theta < 60^\circ \\ 0.1 & 60^\circ \le \theta < 90^\circ \\ 0 & 90^\circ \le \theta \le 180^\circ \end{cases}$$

(a) What is the directivity (above isotropic) of the antenna (in dB)?

(a) $D_0 = \dfrac{4\pi U_{max}}{P_{rad}} = \dfrac{U_{max}}{U_0}$

$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} U \sin\theta \, d\theta = 2\pi \Big\{ \int_0^{30^\circ} \sin\theta \, d\theta +$

$\int_{30^\circ}^{60^\circ} (0.5)\sin\theta \, d\theta + \int_{60^\circ}^{90^\circ} (0.1)\sin\theta \, d\theta \Big\} = 2\pi \Big\{ (-\cos\theta)\Big|_0^{30^\circ} + \left(-\dfrac{\cos\theta}{2}\right)\Big|_{30^\circ}^{60^\circ} + (-0.1\cos\theta)\Big|_{60^\circ}^{90^\circ} \Big\}$

(Cont'd) $= 2\pi \left\{ (-0.866+1) + \left(\dfrac{-0.5+0.866}{2} \right) + \left(\dfrac{-0+0.5}{10} \right) \right\}$

$P_{rad} = 2\pi \{ -0.866 + 1 - 0.25 + 0.433 + 0.05 \} = 2\pi (0.367)$

$= 0.734 \cdot \pi = 2.3059$

$D_0 = \dfrac{1(4\pi)}{2.3059} = 5.4496 = 7.3636 \text{ dB}$

7. The radiation intensity of an antenna is given by
$\mathbf{U(\theta,\varphi)=\cos^4\theta\sin^2\varphi}$, for $0 \le \theta \le \pi/2$ and $0 \le \varphi \le 2\pi$ (i.e., inthe upper half-space). It is zero in the lower half-space.
Find the
(a) Exact directivity (dimensionless and in dB)
(b) Elevation plane half-power beam width (in degrees).

(a) $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\sin^2\phi\, d\phi \cdot \int_0^{\pi/2}\cos^4\theta\sin\theta\, d\theta$

$= (\pi)(\frac{1}{5}) = \frac{\pi}{5}.$

$U_{max} = U(\theta=0°, \phi=\pi/2) = 1.$

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{(\pi/5)} = 20 = 13.0 \text{ dB}$

(b) Elevation Plane : θ varies , ϕ fixed

→ Choose $\phi = \pi/2$.

$U(\theta, \phi=\pi/2) = \cos^4\theta$, $0 \leqslant \theta \leqslant \pi/2$.

$\cos^4\left[\frac{HPBW(el.)}{2}\right] = \frac{1}{2}$

$HPBW(el.) = 2\cdot\cos^{-1}\{\sqrt{0.5}\} = 65.5°$

8. The far-zone electric-field intensity (array factor) of an end-fire two-element array antenna, placed along the z-axis and radiating into free-space, is given by

$$E = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]\frac{e^{-jkr}}{r}, \qquad 0 \le \theta \le \pi$$

Find the directivity using Kraus' approximate formula

(a). $E\Big|_{max} = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]\Big|_{max} = 1$ at $\theta = 0°$.

$0.707\, E_{max} = 0.707\cdot(1) = \cos\left[\frac{\pi}{4}(\cos\theta_1 - 1)\right]$

$\frac{\pi}{4}(\cos\theta_1 - 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(2) = \text{does not exist} \\ \cos^{-1}(0) = 90° = \frac{\pi}{2} \text{ rad.} \end{cases}$

$\Theta_{1r} = \Theta_{2r} = 2(\frac{\pi}{2}) = \pi$

$D_o \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$

9. The normalized far-zone field pattern of an antenna is given by

$$E = \begin{cases} (\sin\theta\cos^2\phi)^{1/2} & 0 \le \theta \le \pi \text{ and } 0 \le \phi \le \pi/2, 3\pi/2 \le \phi \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the directivity using

(a) The exact expression
(b) Kraus' approximate formula

$$U = \frac{1}{2\eta}|E|^2 = \frac{1}{2\eta}\sin\theta\cos^2\phi \Rightarrow U_{max} = \frac{1}{2\eta}$$

(a). $$P_{rad} = 2\cdot\int_0^{\pi/2}\int_0^{\pi}\frac{1}{2\eta}\sin^2\theta\cos^2\phi\, d\theta\, d\phi = \frac{1}{\eta}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8\eta}$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left(\frac{1}{2\eta}\right)}{\frac{\pi^2}{8\eta}} = \frac{16}{\pi} = 5.09 = 7.07\text{ dB}$$

(b). $U_{max} = \frac{1}{2\eta}$ at $\theta = \pi/2$, $\phi = 0$

In the elevation plane through the maximum $\phi = 0$ and $U = \frac{1}{2\eta}\sin\theta$.

The 3-dB point occurs when

$$U = 0.5\,U_{max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}(0.5) = 30^\circ$$

Therefore $\Theta_{1d} = 2(90 - 30) = 120^\circ$

In the azimuth plane through the maximum $\theta = \pi/2$ and $U = \frac{1}{2\eta}\cos^2\phi$.

The 3-dB point occurs when $U = 0.5\,U_{max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\cos^2\theta_1 \Rightarrow$

$$\phi_1 = \cos^{-1}(0.707) = 45^\circ, \quad \Theta_{2d} = 2(90^\circ - 45^\circ) = 90^\circ$$

Therefore using Kraus' formula $D_0 \simeq \frac{41,253}{120\cdot(90)} = 3.82 = 5.82\text{ dB}$

10. A uniform plane wave is traveling in the positive z-direction. Find the polarization (linear, circular, or elliptical), sense of rotation (CW or CCW), axial ratio (AR), and tilt angle τ (in degrees) when

(a) Ex =Ey, Δφ=φy−φx =0
(b) Ex ≠Ey, Δφ=φy−φx =0
(c) Ex =Ey, Δφ=φy−φx =π/2
(d) Ex =Ey, Δφ=φy−φx =−π/2
(e) Ex =Ey, Δφ=φy−φx =π/4
(f) Ex =Ey, Δφ=φy−φx =−π/4
(g) Ex =0.5Ey, Δφ=φy−φx =π/2
(h) Ex =0.5Ey, Δφ=φy−φx =−π/2

Inall cases, justify the answer.

(a) Linear because $\Delta\phi = 0$.

(b) Linear because $\Delta\phi = 0$.

(c) Circular because 1. $E_x = E_y$

2. $\Delta\phi = \pi/2$.

CCW because E_y leads E_x. $AR = 1$, $\tau = 90°$

(d) Circular because 1. $E_x = E_y$

2. $\Delta\phi = -\pi/2$

CW because E_y lags E_x. $AR = 1$, $\tau = 90°$

(e) Elliptical because $\Delta\phi$ is not multiples of $\pi/2$.

CCW because E_y leads E_x. $AR = OA/OB$

Letting $E_x = E_y = E_o$

$$\left.\begin{aligned} OA &= E_o[0.5(1+1+\sqrt{2})]^{1/2} = 1.30656\ E_o \\ OB &= E_o[0.5(1+1-\sqrt{2})]^{1/2} = 0.541196\ E_o \end{aligned}\right\} \Rightarrow AR = \frac{1.30656}{0.541196} = 2.414$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left[\frac{2(1)\cos(45°)}{1-1}\right] = 90° - \frac{1}{2}\tan^{-1}\left(\frac{1.414}{0}\right)$$

$$= 90° - \frac{1}{2}(90°) = 45°$$

(f) Elliptical because $\Delta\phi$ is not multiples of $\pi/2$

CW because E_y lags E_x

From above $\left.\begin{aligned} OA &= 1.30656\ E_o \\ OB &= 0.541196\ E_o \end{aligned}\right\} \Rightarrow AR = \frac{1.30656}{0.541196} = 2.414$

From above $\tau = 90° - \frac{1}{2}(90°) = 45°$

(g). Elliptical because 1. $E_x \neq E_y$

2. $\Delta\phi$ is not zero or multiples of π.

CCW because E_y leads E_x.

$$\left.\begin{aligned} OA &= E_y\left\{\frac{1}{2}[0.25+1+0.75]\right\}^{1/2} = E_y \\ OB &= E_y\left\{\frac{1}{2}[0.25+1-0.75]\right\}^{1/2} = 0.5\ E_y \end{aligned}\right\} \Rightarrow AR = \frac{1}{0.5} = 2.$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left(\frac{0}{-0.75}\right) = 90° - \frac{1}{2}(180°) = 0°$$

(h) Elliptical because 1. $E_x \neq E_y$

2. $\Delta\phi$ is not zero or multiples of π.

CW because E_y lags E_x.

From above $\left.\begin{aligned} OA &= E_y \\ OB &= 0.5\ E_y \end{aligned}\right\} \Rightarrow AR = \frac{1}{0.5} = 2$

$$\tau = 90° - \frac{1}{2}(180°) = 0°.$$

Good Luck

Dr. Gehan Sami